SECTION 4.5: OPTIMIZATION

Name: ____________________

DEFINITIONS: Suppose f is defined on an interval I.

- The value $f(c)$ is called the **absolute maximum** of f on I if $f(x) \leq f(c)$ for all x in I.
- The value $f(c)$ is called the **absolute minimum** of f on I if $f(c) \leq f(x)$ for all x in I.
- The value $f(c)$ is called a **local maximum** if $f(x) \leq f(c)$ for all x in some open interval in I containing c.
- The value $f(c)$ is called a **local minimum** if $f(c) \leq f(x)$ for all x in some open interval in I containing c.

NOTE: The maximum and minimum values of a function are called the '**extreme values**' or the '**extrema**.'

EXTREME VALUE THEOREM: (EVT) If f is **continuous** on $[a, b]$, then f attains its extrema on $[a, b]$. That is, there are values x_1 and x_2 in $[a, b]$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all x in $[a, b]$.

The extreme values of a continuous function on an interval $[a, b]$ could occur at the endpoints, a and/or b, or in the interior, (a, b). If the latter, the extrema are local extrema. The next theorem helps us locate local extrema.

FERMAT'S THEOREM: If $f(c)$ is a local extreme value of f, then $f'(c) = 0$ or $f'(c)$ does not exist.

NOTE: There are examples where $f'(c) = 0$ or $f'(c)$ does not exist and $f(c)$ is **NOT** a local extreme value.[1]

A number c in the domain of the function f is called a **critical value** if $f'(c) = 0$ or $f'(c)$ does not exist.

OPTIMIZING CONTINUOUS FUNCTIONS ON CLOSED BOUNDED INTERVALS:

1. Verify f is continuous on $[a, b]$, so the EVT applies.
2. Find the critical values: use Fermat's Theorem: solve $f'(x) = 0$ or determine where $f'(x)$ does not exist.
3. Evaluate f at the endpoints and the critical values:
   - the largest function value is the maximum of f.
   - the smallest function value is the minimum of f.

If the function we're trying to optimize is continuous, but the interval is not closed or is unbounded, we have:

LONE CRITICAL VALUE: (LCV) If f is **continuous** on an interval I and c is the *only* critical value of f in I:

- if $f(c)$ is a local maximum value, then $f(c)$ is the absolute maximum on I.
- if $f(c)$ is a local minimum value, then $f(c)$ is the absolute minimum on I.

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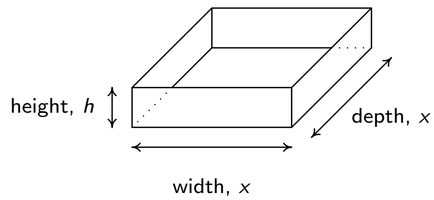
[1] Consider $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ at $x = 0$.

GUIDELINES FOR WORKING APPLIED OPTIMIZATION PROBLEMS

1. Identify the quantity you wish to maximize or minimize. (This is often called the 'objective' function.)
2. Assign variables, as needed, and use formulas given in the problem (or from geometry) to relate the variables to the objective function. It is helpful to carry units along here to make sure the equations make sense at the 'level of units.' For example, you can't add feet to inches and get square feet.
3. Since this is only Calc 1, you will need to get your objective function as a function of just *one* variable. You may need to use some of the formulas from Step 2 to do this.
4. Determine a reasonable applied domain for the problem. For example, if you're asked to build a fence, the length and width of the fence need to be positive quantities.
5. If your applied domain is a closed and bounded interval, use §4.1. Otherwise, shoot for using the LCV. If that doesn't work, use your curve sketching prowess to analyze the function.
6. Check the reasonableness of your answer. If your minimum 'area' is -300 square feet, you either did something really wrong, or have just found something to publish in the Journal of Theoretical Physics.
7. Don't give up! Sometimes you need to abandon one line of thinking completely and start from scratch.

EXAMPLES FROM THE TEXTBOOK: Homework problems: 16, 22, 42, 44

EXAMPLE 1: (VIDEO) A box with a **square** base and no top is to be constructed so that it has a volume of 500 cubic centimeters. Let x denote the width of the box, in centimeters as seen below.



1. Find an expression for the height of the box, h, in terms of x.

   HINT: Use what you know about the volume of the box . . .

   Ans: We know the volume of the box is to be 500 cubic centimeters.

   Since $V = (\text{width}) \cdot (\text{depth}) \cdot (\text{height}) = x^2\, h$ this means $500 = x^2\, h$, so $h = \dfrac{500}{x^2}$.

2. Find the surface area, S, as a function of x. What is a reasonable applied domain?

   Ans: The surface area $S = \text{area of base} + \text{area of sides} = x^2 + 4xh$.

   Since $h = \dfrac{500}{x^2}$, we have $S = x^2 + 4x\left(\dfrac{500}{x^2}\right) = x^2 + \dfrac{2000}{x}$.

   Hence we have $S(x) = x^2 + \dfrac{2000}{x}$. Since x represents a length, we restrict our attention to $x > 0$.

3. Find the dimensions of the box which minimize the surface area.

   Make sure you prove why your dimensions result in the absolute minimum.

   Ans: We find $S'(x) = 2x - \dfrac{2000}{x^2}$. Solving $S'(x) = 0$ gives $2x - \dfrac{2000}{x^2} = 0$.

   We get $2x = \dfrac{2000}{x^2}$ or $x^3 = 1000$, so $x = 10$.

   To show $x = 10$ produces a minimum, we could use the first or second derivative test.

   In this case, we can quickly find $S''(x) = 2 + \dfrac{4000}{x^3}$ so $S''(10) > 0$.

   Hence we have a local minimum when $x = 10$.

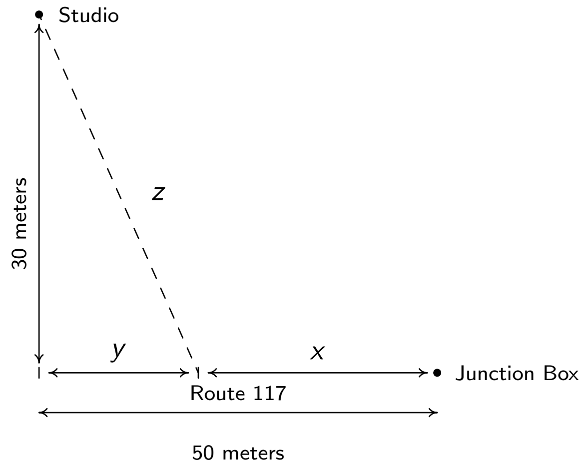
   Since $x = 10$ is the only critical number in our domain, $x > 0$, and we have a local minimum at $x = 10$, we know we must have the absolute minimum at $x = 10$. (This is the LCV in action!)

   Hence the box with minimum surface area has a base of 10 cm by 10 cm.

   To find the height, we use the equation: $h = \dfrac{500}{x^2} = \dfrac{500}{(10)^2} = 5$, so the height of the box is 5 cm.

   NOTE: The minimum surface area is $S(10) = (10)^2 + \dfrac{2000}{10} = 100 + 200 = 300$ square centimeters.

EXAMPLE 2: (VIDEO) CouchJockey wishes to get high speed internet service installed in his new streaming studio located 30 meters from Route 117. The nearest junction box is located 50 meters down road from the post, as indicated in the diagram below. Suppose it costs \$1.50 per meter to run cable along the road and \$2.50 per meter to run cable off of the road. Determine how far along Route 117 the cable should be run from the Junction Box before turning off road, x, so that the total cost of running the cable is minimized.



1. Express the distance off-road, z, as a function of x.

   Ans: From the Pythagorean Theorem, we have $z^2 = y^2 + 30^2 = y^2 + 900$.

   Since $x + y = 50$, we have $y = 50 - x$ so $z^2 = (50 - x)^2 + 900$ or $z = \sqrt{(50 - x)^2 + 900}$.

2. Express the total cost of running the cable, C, as a function of x. What is a reasonable applied domain?

   Ans:

$$\begin{aligned} C &= \text{cost along Route 117} + \text{cost off-road} \\ &= \$1.50(\text{distance along Route 117}) + \$2.50(\text{distance off-road}) \\ C(x) &= 1.5x + 2.5\sqrt{(50 - x)^2 + 900} \end{aligned}$$

   For our domain, we note that $0 \le x \le 50$, with $x = 0$ corresponding to running all the internet off-road and $x = 50$ corresponding to running internet the entire length of Route 117.

3. Determine how far along Route 117 the cable to be run so as to minimize the cost.

   Ans: Since C is a continuous function defined over a closed, bounded interval $[0, 50]$, all we need to do to optimize C is to find the critical numbers in $(0, 50)$ and compare the the value of C at the critical numbers with $C(0)$ and $C(50)$. To that end, we first find $C'(x)$.

$$
\begin{aligned}
C'(x) &= D_x\left[1.5x + 2.5\sqrt{(50-x)^2+900}\right] \\
&= 1.5 + 2.5\left(\frac{1}{2}\right)\left[(50-x)^2+900\right]^{-\frac{1}{2}} D_x\left[(50-x)^2+900\right] \\
&= 1.5 + 1.25\left[(50-x)^2+900\right]^{-\frac{1}{2}}[2(50-x)]D_x[(50-x)] \\
&= 1.5 + 2.5\left[(50-x)^2+900\right]^{-\frac{1}{2}}(50-x)(-1) \\
C'(x) &= 1.5 - \frac{2.5(50-x)}{\sqrt{(50-x)^2+900}}
\end{aligned}
$$

Solving $C'(x) = 0$ gives :

$$
\begin{aligned}
C'(x) &= 0 \\
1.5 - \frac{2.5(50-x)}{\sqrt{(50-x)^2+900}} &= 0 \\
\frac{2.5(50-x)}{\sqrt{(50-x)^2+900}} &= 1.5 \\
2.5(50-x) &= 1.5\sqrt{(50-x)^2+900} \\
5(50-x) &= 3\sqrt{(50-x)^2+900} \\
25(50-x)^2 &= 9\left[(50-x)^2+900\right] \\
25(50-x)^2 &= 9(50-x)^2+8100 \\
16(50-x)^2 &= 8100 \\
(50-x)^2 &= \frac{8100}{16} \\
50-x &= \pm\frac{90}{4} = \pm 22.5
\end{aligned}
$$

From $50 - x = \pm 22.5$, we get $x = 27.5$ and $x = 72.5$. Only $x = 27.5$ lies in the interval $(0, 50)$.

We compute: $C(0) \approx 145.77$, $C(27.5) = 135$, and $C(50) = 150$. Hence to minimize cost, we run the internet 27.5 meters along Route 117 then turn off-road for a cost of \$135.00.

HOMEWORK: Section 4.5: 13 - 49 every other odd; 54*, 55*, 60*, 64*